Problem 1

(7 pts each)Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = \left(\dfrac{n-3}{n+3}\right)^n$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\left(\frac{n\left(1-\frac{3}{n}\right)}{n\left(1+\frac{3}{n}\right)}\right)^n = \lim_{n\to+\infty}\frac{\left(1-\frac{3}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$= \frac{e^{-3}}{e^3} = e^{-6} \checkmark$$

$\textcircled{7}$

(b) $b_n = \dfrac{9^n \sin\left(4^n+n^9+10\right)}{n!}$

$$-1 \le \sin\left(4^n+n^9+10\right) \le 1$$

$$-9^n \le 9^n\sin\left(4^n+n^9+10\right) \le 9^n$$

$$\frac{-9^n}{n!} \le \frac{9^n\sin\left(4^n+n^9+10\right)}{n!} \le \frac{9^n}{n!}$$

$\textcircled{7}$

$$\lim_{n\to+\infty}\frac{-9^n}{n!} = \lim_{n\to+\infty}\frac{9^n}{n!} = 0 \quad \text{(Basic limit)}$$

So by Sandwich Theorem,

$$\lim_{n\to+\infty}\frac{9^n\sin\left(4^n+9^n+10\right)}{n!} = 0 \checkmark$$

(c) $c_n = \dfrac{n^{\frac{1}{\ln n}}(n+1)^7+(\ln n)^2}{n^7}$

$$\lim_{+\infty} \frac{n^{\frac{1}{\ln n}}}{n^7} \textcircled{+} \frac{(n+1)^7}{n^7} + \frac{(\ln n)^2}{n^7}$$

$\textcircled{4}$

$$\lim_{+\infty} \frac{e^{\ln n^{\frac{1}{\ln n}}}}{n^7} + \left(\frac{n+1}{n}\right)^7 + \frac{\ln n^2}{n^7}$$

$$\frac{e^{\frac{\ln n}{\ln n}}}{\boxed{n^7}} \checkmark + 1 \checkmark + 0$$ ✓ HO? (because since $\lim_{+\infty}\frac{\ln n}{n} = 0$)

$\downarrow$

0

$= \boxed{1}$

Problem 2

(9 pts each) Which of the following series converge, and which diverge? Find the sum of the series when possible.

(a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3^n}+\frac{2^{n-1}}{5^n}\right)$

$$\sum_{m=0}\left(\frac{-1}{3}\right)^m+\sum_{m=0}^{\infty}\frac{2^m\times 2^{-1}}{5^m}=\overbrace{\sum_{m=0}^{\infty}\left(\frac{-1}{3}\right)^m}^{b_m}+\overbrace{\sum_{m=0}^{\infty}\frac{1}{2}\times\left(\frac{2}{5}\right)^m}^{c_m}$$

- $\sum_{m=0}^{\infty} b_m$ is a geometric serie of ratio $|r|<1$

$$\sum_{m=0}^{\infty}\left(\frac{-1}{3}\right)^m=\frac{1}{1+\frac{1}{3}}=\frac{3}{4}$$

- $\sum_{m=0}^{\infty} c_m$ is a geometric serie of ratio $|r|<1$

$$\frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{2}{5}\right)^m=\frac{1}{2}\times\frac{1}{1-\frac{2}{5}}=\frac{1}{2}\times\frac{5}{3}=\frac{5}{6}$$

By the Algebric properties of series $\sum_{m=0}^{\infty}\frac{(-1)^m}{3^m}+\frac{2^{m-1}}{5^m}$ converges and it is equal to $\frac{3}{4}+\frac{5}{6}=\frac{19}{12}$

(b) $\sum_{n=1}^{\infty}\frac{1}{n^{0.1}\left(2^n+1\right)}$

$$a_m=\frac{1}{m^{0,1}\times(2^m+1)}$$

$$\sum_{m=1}^{\infty} b_m=\sum_{m=1}^{\infty}\frac{1}{m^{1,1}}$$

$$\lim_{m\to+\infty}\frac{a_m}{b_m}=\lim_{m\to+\infty}\frac{m^{1,1}}{m^{0,1}\times(2^m+1)}=\lim_{m\to+\infty}\frac{m}{2^m+1}=\frac{\infty}{\infty}$$

Hopital's Rule: $\lim_{m\to+\infty}\frac{1}{\ln 2\times 2^m}=0$

Also we have $\sum_{m=1}^{\infty} b_m$ converge (p-series, $p=1,1>1$) $\Rightarrow$ $\sum_{m=1}^{\infty} a_m$ converges by limit comparison Test

(c) $\sum_{n=1}^{\infty}\frac{1}{n^{0.9}}ln\left(1+\frac{1}{n^{0.2}}\right)$

$$a_m=\frac{1}{m^{0,9}}\times\ln\left(1+\frac{1}{m^{0,2}}\right)$$

$$\sum_{m=1}^{\infty} b_m=\sum_{m=1}^{\infty}\frac{1}{m^{1,1}}$$

$$\lim_{m\to+\infty}\frac{a_m}{b_m}=\lim_{+\infty}\frac{\ln\left(1+\frac{1}{m^{0,2}}\right)\times m^{1,1}}{m^{0,9}}=\lim_{+\infty}\ln\left(1+\frac{1}{m^{0,2}}\right)\times m^{0,2}$$

$$=\lim_{m\to+\infty}\frac{\ln\left(1+\frac{1}{m^{0,2}}\right)}{\frac{1}{m^{0,2}}}=1\quad(\text{basic limit})$$

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$\downarrow$ $\frac{1}{m^{0,2}}\to 0$

And since $\sum_{m=1}^{\infty} b_m$ converge, $\sum_{m=1}^{\infty} a_m$ converge by limit comparison test

Problem 3

(14 pts) Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n^{1.4}} (x-5)^n .$$

(Remember to check the endpoints.)

$$|a_m| = \left| \frac{\ln m}{m^{1,4}} \right| \times |x-5|^m$$

$$\lim_{m\to+\infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to+\infty} \left| \frac{\ln(m+1)}{(m+1)^{1,4}} \times \frac{m^{1,4}}{\ln(m)} \right| \times \frac{|x-5|^{m+1}}{|x-5|^m}$$

$$= \lim_{+\infty} \left| \boxed{\frac{\ln(m+1)}{\ln(m)}} \times \boxed{\left(\frac{m}{m+1}\right)}^{1,4} \right| \times |x-5|$$

$$\Longrightarrow \quad \xrightarrow{\text{Hopital's Rule } \left(\frac{\infty}{\infty}\right)} \qquad \downarrow 1 \qquad 4$$

$$\lim_{+\infty} \frac{\frac{1}{m+1}}{\frac{1}{m}} \times |x-5| = \lim_{+\infty} \frac{m}{m+1} |x-5| = |x-5|$$

If $\sum_{m=2}^{\infty} a_m$ converges $\Rightarrow$ $\left|\frac{a_{m+1}}{a_m}\right| < 1$

$|x-5| < 1$ 2

radius of convergence : $R = 1$ 2

interval of convergence : $-R+5 < x < R+5$

$4 < x < 6$ 2

for $x = 4$ $\Rightarrow$ $\sum_{m=2}^{\infty} \frac{(-1)^m \times \ln m}{m^{1,4}} \times (-1)^m = \sum_{m=2}^{\infty} \frac{\ln m}{m^{1,4}}$

for $m \geq 3$ $\quad 1 \leq \ln m$

$$\frac{1}{m^{1,4}} \leq \frac{\ln m}{m^{1,4}}$$

$\sum_{m=3}^{\infty} \frac{1}{m^{1,4}}$ converges (p-series $p = 1,4 > 1$) $\rightarrow$ $\sum_{m=3}^{\infty} \frac{\ln m}{m^{1,4}}$ converges

by direct comparison test $\rightarrow$ $\sum_{m=2}^{\infty} \frac{\ln m}{m^{1,4}} = a_2 + \sum_{m=3}^{\infty} \frac{\ln m}{m^{1,4}}$

converges $\longrightarrow$

for $x = 6$: $\sum_{n=2}^{\infty} \underbrace{\frac{(-1)^n * \ln n}{n^{1.14}}}_{b_n}$ ✓

$\sum_{n=2}^{\infty} |b_n| = \sum_{n=2}^{\infty} \frac{\ln n}{n^{1.14}}$ but it converges (proved already)

① So by absolute convergence test $\sum_{n=2}^{\infty} b_n$ converges

interval of convergence: $4 \le x \le 6$

②

Problem 4

(a) (6 pts) Use the integral test to prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ diverges.

$a_n = f(n)$ with $f(x) = \frac{1}{x \ln x}$

$f(x)$ is : • positive $\left(\begin{array}{l} \ln x > 0 \\ x > 0 \end{array} \text{ for } x \geq 2 \right)$

• continuous $\left(\begin{array}{l} x \ln x \neq 0 \\ x \neq 0 \text{ or } \ln x \neq 0 \\ \qquad\qquad x \neq 1 \\ \text{for } x \geq 2 \text{ it is continuous} \end{array} \right)$

• decreasing

$$f'(x) = \frac{-\ln x - \frac{1}{x} \times x}{(x \ln x)^2} = \frac{-\ln x - 1}{(x \ln x)^2}$$ ✓

$-\ln x - 1 = 0$

$\ln x = -1$

$x = e^{-1}$

$$\begin{array}{c|cccc} x & 0 & e^{-1} & 2 & +\infty \\ \hline -1-\ln x & + & | & - & \end{array}$$

$f'(x) \leq 0$ for $x \geq 2 \Rightarrow f(x)$ decreasing for $x \geq 2$

By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ and $\int_2^{\infty} \frac{1\,dx}{x \ln x}$ both converges or diverges

$$\lim_{R \to +\infty} \int_2^{\cancel{\infty}R} \frac{\frac{1}{x}}{\ln x}\, dx \overset{\lim_{+\infty}}{=} \int_2^R \frac{(\ln x)'}{\ln x}\, dx$$

$$= \left[\ln(\ln x) \right]_2^R = \lim_{R \to +\infty} \ln \ln R - \ln \ln 2$$

$$= +\infty - \ln \ln 2 = +\infty$$

It diverges By the integral test

(b) (4 pts) Decide if $\sum_{n=2}^{\infty} \frac{1}{ln(n!)}$ converges or diverges.

$\ln(n!) < n$ Since $\frac{\ln n}{n} = 0$ (basic limit)

$\frac{1}{\ln(n!)} > \frac{1}{n}$

But $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series $p = 1$) so it diverges by direct comparison test

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(c) (6 pts)Use the alternating series estimation theorem (ASET) to estimate $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(ln\,n)}$ with an error of magnitude no greater than $\frac{1}{5(ln\,5)}$.
Decide if your estimate is an over-estimate or an under-estimate.

$U_n = \frac{1}{n \ln n}$ we just proved that, U_n positive $n \geq 2$

U_n decreasing $n \geq 2$

$\lim_{n \to +\infty} U_n = \frac{1}{+\infty} = 0$

So by alternating series theorem, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ converges

by ASET, |error| < first unused term

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = \frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \frac{1}{5\ln 5} \Big| + \frac{1}{6\ln 6} \;....$$

with |error| < |first unused term|

$$|error| < \left|\frac{1}{6\ln 6}\right| < \frac{1}{5\ln 5} \text{ as required}$$

and since the first unused term is positive $\Rightarrow$ error > 0

$\Rightarrow$ my estimate is under-estimate

Problem 5

(a)(6 pts) Use the fact that $\left(ln(1+x)\right)' = \dfrac{1}{1+x}$ to find a power series expansion for the function $f(x) = ln(1+x)$ about the center $a = 0$.

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^m = \sum_{m=0}^{\infty} x^m$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + x^m = \sum_{m=0}^{\infty} (-1)^m x^m$$

$$\text{Integrate } \int \frac{1}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1}$$

$$\ln(1+x) + C = \sum_{m=0}^{\infty} (-1)^m \times \frac{x^{m+1}}{m+1}$$

$$x = 0, \quad \ln(1) + C = 0$$

$$C = 0$$

$$\ln(1+x) = \sum_{m=0}^{\infty} (-1)^m \times \frac{x^{m+1}}{m+1}$$

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(b) (5 pts) Use part (a) to find $\sum_{n=1}^{\infty} \dfrac{(0.1)^n}{n}$.

$$x = 0,1 \qquad \sum_{m=1}^{\infty} \frac{(-1)^m (0,1)^m}{m} = \sum_{m=0}^{\infty} \frac{(0,1)^{m+1} \times (-1)^{m+1}}{m+1}$$

$$\ln(1+x) = \sum_{m=0}^{\infty} (-1)^m \times \frac{x^{m+1}}{m+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1-0,1) = \sum_{m=0}^{\infty} (-1)^m \times \frac{(0,1)^{m+1} \times (-1)^{m+1}}{m+1}$$

$$\ln(0,9) = -\sum_{m=0}^{\infty} \frac{(0,1)^{m+1}}{m+1}$$

$$\ln(0,9) = 0,1 + \sum_{m=1}^{\infty} \frac{(0,1)^{m+1}}{m+1}$$

$$\ln(0,9) - 0,1 = \sum_{m=1}^{\infty} \frac{(0,1)^m}{m+1}$$

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Problem 6

(a)(7 pts) State and prove the *n*th term test.

State: if $\sum a_m$ converge $\longrightarrow \lim\limits_{+\infty} a_m = 0$

or equivalently if $\lim\limits_{+\infty} a_m \neq 0 \rightarrow \sum a_n$ diverge

Proof: Assume that $\sum a_m$ converges, so we can conclude that

$\lim\limits_{+\infty} S_m$ exists $(S_m = a_1 + a_2 + \dots + a_m)$

Assume that $\lim\limits_{+\infty} S_m = \ell$

$$\left.\begin{array}{l} S_1 = a_1 \\ S_2 = a_2 - a_1 \end{array}\right\} \begin{array}{l} S_2 - S_1 = a_2 \\ \Rightarrow S_3 - S_2 = a_3 \\ \vdots \quad \vdots \\ S_m - S_{m-1} = a_m \end{array}$$

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$$\lim_{+\infty} a_m = \lim_{+\infty} S_m - \lim_{+\infty} S_{m-1} = \ell - \ell = 0$$

(b)(4 pts) Decide if $\sum\limits_{n=1}^{\infty} \left(4^n - 3^n\right)^{1/n}$ converges or diverges.

$\leq 4^m - 3^m \leq 4^m$

0